

# Parametric Cutoffs for Interacting Fermi Liquids

M. Disertori<sup>1</sup>, J. Magnen<sup>2</sup> and V. Rivasseau<sup>3</sup>

1) Laboratoire de Mathématiques Raphaël Salem  
CNRS UMR 6085, Université de Rouen, 76801 Rouen Cedex

2) Centre de Physique Théorique, CNRS UMR 7644,  
Ecole Polytechnique F-91128 Palaiseau Cedex, France

3) Laboratoire de Physique Théorique, CNRS UMR 8627,  
Université Paris XI, F-91405 Orsay Cedex, France

January 21, 2013

## Abstract

This paper is a sequel to [1]. We introduce a new multiscale decomposition of the Fermi propagator based on its parametric representation. We prove that the corresponding sliced propagator obeys the same direct space bounds than the decomposition used in [1]. Therefore the non perturbative bounds on completely convergent contributions of [1] still hold. In addition the new slicing better preserves momenta, hence should become an important new technical tool for the rigorous analysis of condensed matter systems. In particular it should allow to complete the proof that a three dimensional interacting system of Fermions with spherical Fermi surface is a Fermi liquid in the sense of Salmhofer's criterion.

## 1 Introduction

Interacting Fermi liquid theory is not valid down to zero temperature. Below some critical temperature the quasi-particles with momenta near the Fermi surface bound into Cooper pairs. This generic phenomenon goes under the

name of Kohn-Luttinger instabilities. Hence the mathematical definition of Fermi liquid behavior is not obvious.

There are essentially two main ways to block the formation of Cooper pairs, namely to increase temperature or magnetic field.

With a generic strong magnetic field, parity invariance of the Fermi surface is broken and a true discontinuity at a well-defined Fermi surface may be proved mathematically. This is the road followed by Feldman, Knörrer and Trubowitz in the impressive series of papers [2], in which they proved two dimensional Fermi liquid behavior at zero temperature for sufficiently convex and regular parity-breaking Fermi surfaces.

Magnetic fields responsible for parity breaking are also the source of the quantum Hall effect. The rigorous treatment of this effect could require a non-commutative formulation of renormalization group [3] and a suitable generalization of the parametric cutoffs of the present paper.

A criterion to characterize Fermi liquid behavior without breaking parity has been proposed in [4]. Salmhofer remarked that staying in a domain  $|\lambda \log T| \leq K$ , where  $\lambda$  is the coupling constant and  $T$  is the temperature, avoids Cooper pairing and Kohn-Luttinger singularities. Therefore after mass renormalization of the two point function Schwinger functions should be analytic in  $\lambda$  in such a domain  $|\lambda \log T| \leq K$ . Salmhofer criterion states that for Fermi liquids the self-energy and its first and second momentum derivatives remain uniformly bounded in such a domain.

Fermionic models in one dimension are Luttinger liquids [5, 6, 7] and they do not obey the Salmhofer criterion. In two dimensions it has been proved that interacting Fermion systems with a circular [8] or approximately circular [9] Fermi surface obey this criterion. In contrast the Hubbard model at half filling, which has a square Fermi surface, violates the criterion and its self-energy behaves as a Luttinger liquid with logarithmic corrections [10]. Hence Salmhofer criterion effectively distinguishes Fermi-like liquids from Luttinger-like liquids in two dimensions. All these results rely on the special momentum conservation rules of interacting Fermi systems in two dimensions.

In three dimensions Fermi liquid behavior is generically expected but momentum conservation rules allow for non planar vertices and the two-dimensional methods do not extend to the constructive level. The only existing constructive method has been pioneered in [11] and further developed in [1]. It relies on a direct space decomposition of the propagator combined with cluster expansions and Hadamard's inequalities. This is a bit surprising for a constructive Fermionic problem, which can usually be treated with

Gram's inequalities and no cluster expansions [12, 13]. For a recent pedagogical introduction to these questions and further explanations see [14].

It was proved in [1] that the sum of all convergent contributions is indeed analytic in a Salmhofer domain. However the mass renormalization (which in this context can be interpreted as a change of the Fermi surface radius) was not performed, and Salmhofer's criterion was not checked, because the cutoffs used on the propagator did not conserve momentum, hence the computation of the self-energy (i.e. the one-particle irreducible (1PI) amputated two point function) was not automatic in this formalism. Extracting the 1PI two point function would have required a sequence of Mayer expansions to remove hard-core constraints in the cluster expansion.

The situation remained in this incomplete stage for a decade. In this paper we define a new slice decomposition of the propagator at finite temperature which approximately conserves momentum. Our main result is Theorem 1 in Section 4 which states that this new slicing obeys the spatial bounds of the former slicing used in [1]. Its proof relies on a saddle point analysis with rigorous control of the remainder terms.

As a consequence the bounds of [1] on convergent contributions which do not require mass renormalization also hold for this new decomposition, but mass renormalization of the two point divergent subgraphs should become much easier. Indeed momentum preserving cutoffs have the nice property that two point subgraphs made of higher slices than their external legs are automatically one particle irreducible. This was the key to simplify their renormalization and to prove Salmhofer's bounds in all previous works [8, 9, 10].

It is therefore likely that using this new propagator slicing the program of [1] can be completed, although the mass renormalization and the complete proof of Salmhofer's criterion remain beyond the scope of the present paper.

## 2 The model

The model is the isotropic jellium in three spatial dimensions with a local four point interaction considered in [1]. We recall for completeness the corresponding notations and conventions.

## 2.1 Free propagator

Using the Matsubara formalism, the propagator in Fourier space  $\hat{C}$  is equal to:

$$\hat{C}_{ab}(k_0, k) = \delta_{ab} \frac{1}{ik_0 - e(k)}, \quad e(k) = \frac{k^2}{2m} - \mu, \quad (2.1)$$

where  $a, b \in \{\uparrow, \downarrow\}$  are the spin indices. The vector  $k$  in (2.1) is three-dimensional. The parameters  $m$  and  $\mu$  correspond to the effective mass and to the chemical potential (which fixes the Fermi energy). To simplify notation we put  $2m = \mu = 1$ , so that  $e(k) = k^2 - 1$ . The corresponding direct space propagator at temperature  $T$  and position  $(t, x)$  (where  $x$  is the three dimensional spatial component) is

$$C_{ab}(t, x) = \frac{T}{(2\pi)^3} \sum_{k_0} \int d^3k e^{-ik_0 t + ikx} \hat{C}_{ab}(k_0, k), \quad (2.2)$$

and is antiperiodic in the variable  $t$  with antiperiod  $\frac{1}{T}$ . This means that

$$\hat{C}(k_0, k) = \frac{1}{2} \int_{-\frac{1}{T}}^{\frac{1}{T}} dt \int d^3x e^{+ik_0 t - ikx} C(t, x) \quad (2.3)$$

is not zero only for discrete values (called the Matsubara frequencies) :

$$k_0 = (2n + 1)\pi T, \quad n \in \mathbb{Z}, \quad (2.4)$$

where we take  $\hbar = k = 1$ . Remark that only odd frequencies appear, because of antiperiodicity, hence  $|k_0| \geq \pi T$  so that the temperature acts like an effective infrared cutoff.

The notation  $\sum_{k_0}$  in (2.2) means really the discrete sum over the integer  $n$  in (2.4)<sup>1</sup>. To simplify notations we write:

$$\int d^4k \equiv T \sum_{k_0} \int d^3k, \quad \int d^4x \equiv \frac{1}{2} \int_{-1/T}^{1/T} dt \int d^3x. \quad (2.5)$$

---

<sup>1</sup>When  $T \rightarrow 0$ ,  $k_0$  becomes a continuous variable, the discrete sum becomes an integral  $T \sum_{k_0} \rightarrow \frac{1}{2\pi} \int dk_0$ , and the corresponding propagator  $C_0(k_0, k)$  becomes singular on the Fermi surface defined by  $k_0 = 0$  and  $|k| = 1$ .

## 2.2 Propagator with ultraviolet cutoff

We remember that we can add a continuous ultraviolet cut-off (at a fixed scale  $\Lambda_u = 1$ ) to the propagator (2.1). For convenience we introduced this cutoff both on spatial and on Matsubara frequencies; indeed the Matsubara cutoff could be lifted with little additional work. The propagator (2.1) equipped with this cut-off is called  $C^u$  and is defined as:

$$\hat{C}^u(k_0, k) := \frac{e^{-[k_0^2 + (k^2 - 1)^2]}}{ik_0 + (k^2 - 1)} . \quad (2.6)$$

Note that in previous works we used a compact support function for this ultraviolet cutoff. Here we use an exponential because it is better adapted to the parametric representation that we shall use.

## 2.3 Partition function

Finally we introduce the local four point interaction

$$I(\psi, \bar{\psi}) = \lambda \int_{\Lambda} d^4x (\bar{\psi}_{\uparrow} \psi_{\uparrow})(\bar{\psi}_{\downarrow} \psi_{\downarrow}) = \lambda \int_{\Lambda} d^4x \prod_{c=1}^4 \psi_c , \quad (2.7)$$

where  $\psi_c$  is defined as:

$$\psi_1 = \bar{\psi}_{\uparrow} \quad \psi_2 = \psi_{\uparrow} \quad \psi_3 = \bar{\psi}_{\downarrow} \quad \psi_4 = \psi_{\downarrow} \quad (2.8)$$

The partition function is then defined as

$$\begin{aligned} Z_{\Lambda}^u &= \int d\mu_{C^u}(\psi, \bar{\psi}) e^{I(\psi, \bar{\psi})} = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\mu_{C^u}(\psi, \bar{\psi}) I(\psi, \bar{\psi})^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int d\mu_{C^u}(\psi, \bar{\psi}) \prod_{v \in V} I_v(\psi, \bar{\psi}) \end{aligned} \quad (2.9)$$

where  $V$  is the set of  $n$  vertices and  $I_v(\psi, \bar{\psi})$  denotes the local interaction at vertex  $v$ .

In order to perform a multiscale analysis we need to introduce a slice decomposition over fields. This corresponds to introduce a slice decomposition on the free propagator (with UV cutoff)  $C^u$ .

### 3 The propagator

In [1] we introduced a multiscale analysis directly on position space. It is more convenient to introduce a new scale decomposition which is compatible with momentum conservation. This conservation is indeed useful to control renormalization of two point functions.

This new decomposition is the main technical innovation of this paper with respect to [1]. It cuts slices on the integration range of the Schwinger parameter: this a good compromise between  $x$  and  $p$  space slicing.

#### 3.1 Schwinger representation of the propagator

**Lemma 1** *The propagator (with UV cutoff)  $C^u$  defined above (2.6) can be written as*

$$\hat{C}(k_0, \vec{k}) = \int_1^\infty d\alpha \int_{\mathbb{R}} d\beta \hat{F}(\alpha, \beta, k_0, \vec{k}) \quad (3.10)$$

$$C(t, \vec{x}) = \int_1^\infty d\alpha \int_{\mathbb{R}} d\beta F(\alpha, \beta, t, \vec{x}) \quad (3.11)$$

where

$$\hat{F}(\alpha, \beta, k_0, \vec{k}) = [-ik_0 + (k^2 - 1)] \frac{1}{2\sqrt{\pi\alpha}} e^{-\alpha k_0^2} e^{-\frac{\beta^2}{4\alpha} + i\beta(k^2 - 1)} \quad (3.12)$$

$$F(\alpha, \beta, t, \vec{x}) = -\frac{\sqrt{\pi}}{2(2\pi)^4} \frac{\tilde{I}(\alpha, t) + i\beta I(\alpha, t)}{\alpha^2} e^{-\frac{\beta^2}{4\alpha}} \int d^3k e^{+ik \cdot x} e^{i\beta(k^2 - 1)} \quad (3.13)$$

$$= -\frac{e^{i\frac{3\pi}{4}}}{2(4\pi)^2} \frac{\tilde{I}(\alpha, t) + i\beta I(\alpha, t)}{\alpha^2 \beta^{3/2}} e^{-\frac{\beta^2}{4\alpha}} e^{-iB(\beta, x)}, \quad (3.14)$$

we do not write the  $u$  index for simplicity and we defined

$$B(\beta, x) = \left( \beta + \frac{x^2}{4\beta} \right) \quad (3.15)$$

$$I(\alpha, t) = (\sqrt{\alpha}T) \sum_{k_0} e^{-ik_0 t} e^{-\alpha k_0^2} \quad \tilde{I}(\alpha, t) = -2\alpha \partial_t I(\alpha, t). \quad (3.16)$$

Moreover for  $\alpha < T^{-2}$  and for any  $p > 0$ ,

$$|I| \leq \frac{K_p}{\left(1 + \frac{f(t)}{\sqrt{\alpha}}\right)^p}, \quad \tilde{I} \leq \sqrt{\alpha} \frac{K_p}{\left(1 + \frac{f(t)}{\sqrt{\alpha}}\right)^p} \quad (3.17)$$

and when  $\alpha \geq T^{-2}$

$$|I| \leq \frac{c}{1+Tf(t)} \leq K, \quad \tilde{I} \leq \frac{c}{T} \frac{1}{1+Tf(t)} \leq \frac{K}{T}, \quad (3.18)$$

where  $K_p$  is a constant depending only of  $p$ ,  $K$  and  $c$  are constants and  $f(t)$  is defined by

$$f(t) = \left| \frac{\sin(2\pi Tt)}{2\pi T} \right|, \quad t \in \left[ -\frac{1}{T}, \frac{1}{T} \right] \quad (3.19)$$

**Remark 1** Note that  $f(t) \leq 1/T$  hence the bound in (3.18).

**Remark 2** The function  $I(\alpha, t)$  is a discrete Fourier transform. In the continuum limit we have

$$I(\alpha, t) = \frac{1}{2\sqrt{\pi}} e^{-\frac{t^2}{4\alpha}} \quad \tilde{I}(\alpha, t) = \frac{1}{2\sqrt{\pi}} t e^{-\frac{t^2}{4\alpha}} \quad (3.20)$$

so  $t$  cannot be larger than  $\sqrt{\alpha}$ . This should be true also for  $T$  finite, but instead of exponential we may expect only polynomial decay (3.17). Moreover, as  $|t| \leq 1/T$  we get a decay not directly for  $|t|$  but for  $f(t)$  (3.19), which is what we obtained also for the full propagator in  $x$  space (see [1], section II.3). As in that case, the proof will be based on integration by parts.

**Proof of the first part: (3.10) and (3.11)** The Schwinger representation of the propagator (with its UV cutoff) is

$$\hat{C}(k_0, \vec{k}) = [-ik_0 + (k^2 - 1)] \int_1^\infty d\alpha e^{-\alpha[k_0^2 + (k^2 - 1)^2]} \quad (3.21)$$

We rewrite the quartic term in the exponent as a Gaussian integral

$$e^{-\alpha(k^2 - 1)^2} = \frac{1}{2\sqrt{\pi\alpha}} \int_{\mathbb{R}} d\beta e^{-\frac{\beta^2}{4\alpha} + i\beta(k^2 - 1)}, \quad (3.22)$$

so the propagator becomes

$$\hat{C}(k_0, k) = [-ik_0 + (k^2 - 1)] \int_1^\infty d\alpha \frac{e^{-\alpha k_0^2}}{2\sqrt{\pi\alpha}} \int d\beta e^{-\frac{\beta^2}{4\alpha} + i\beta(k^2 - 1)}. \quad (3.23)$$

In order to get the  $x$  behavior we need to take the Fourier transform

$$C(t, x) = \frac{1}{(2\pi)^3} \int d^4k \, e^{-ik_0 t + ik \cdot x} \hat{C}(k_0, k) \quad (3.24)$$

Now using the following relations<sup>2</sup>

$$\begin{aligned} \int d^3k \, e^{+ik \cdot x} e^{i\beta(k^2-1)} &= e^{-i(\beta + \frac{x^2}{4\beta})} \left( e^{i\frac{\pi}{4}} \sqrt{\frac{\pi}{\beta}} \right)^3 \\ [-ik_0 + (k^2 - 1)] e^{-ik_0 t + i\beta(k^2-1)} &= [\partial_t - i\partial_\beta] e^{-ik_0 t + i\beta(k^2-1)} \end{aligned} \quad (3.25)$$

and the definition (3.16) we get

$$\begin{aligned} C(t, x) &= \frac{1}{(2\pi)^3} \int_1^\infty \frac{d\alpha}{2\alpha\sqrt{\pi}} \int_{\mathbb{R}} d\beta \, e^{-\frac{\beta^2}{4\alpha}} [\partial_t - i\partial_\beta] \\ &\quad \left[ I(\alpha, t) e^{-i(\beta + \frac{x^2}{4\beta})} \left( e^{i\frac{\pi}{4}} \sqrt{\frac{\pi}{\beta}} \right)^3 \right] \\ &= \frac{1}{(4\pi)^2} \int_1^\infty \frac{d\alpha}{\alpha} \int_{\mathbb{R}} \frac{d\beta}{\beta^{3/2}} e^{-i(-3\frac{\pi}{4} + \beta + \frac{x^2}{4\beta})} [\partial_t I + iI\partial_\beta] e^{-\frac{\beta^2}{4\alpha}} \\ &= -\frac{e^{i\frac{3\pi}{4}}}{(4\pi)^2} \int_1^\infty \frac{d\alpha}{\alpha} \int_{\mathbb{R}} \frac{d\beta}{\beta^{3/2}} e^{-i(\beta + \frac{x^2}{4\beta})} e^{-\frac{\beta^2}{4\alpha}} \left( \frac{\tilde{I} + i\beta I}{2\alpha} \right) \\ &= -\frac{e^{i\frac{3\pi}{4}}}{2(4\pi)^2} \int_1^\infty d\alpha \int_{\mathbb{R}} d\beta \, \frac{\tilde{I} + i\beta I}{\alpha^2 \beta^{3/2}} e^{-\frac{\beta^2}{4\alpha}} e^{-iB} \end{aligned} \quad (3.26)$$

where in the second line we applied integration by parts with respect to  $\beta$ , and the definitions (3.16) for  $\tilde{I}$  and (3.15) for  $A, B$ . Note that

$$\begin{aligned} |I| &\leq \frac{\sqrt{\alpha}}{(2\pi)} \int dk_0 \, e^{-\alpha k_0^2} = \frac{1}{2\sqrt{\pi}} \\ |\tilde{I}| &\leq 2 \frac{\alpha^{3/2}}{(2\pi)} \int dk_0 \, |k_0| e^{-\alpha k_0^2} = \frac{\sqrt{\alpha}}{\pi} . \end{aligned} \quad (3.27)$$

Therefore the final expression is integrable separately in  $\alpha$  and  $\beta$  and the integration order no longer matters. This completes the proof of (3.10) and (3.11). If we do not perform explicitly the Fourier transform with respect to  $\vec{k}$  we get (3.13).  $\square$

---

<sup>2</sup>To prove the first relation: the linear term can be eliminated by a real translation; the remaining integral can be computed on the positive real semi-axis by rotating the contour in the complex plane by an angle  $\pi/4$ .



**Proof of the second part: (3.17) and (3.18)** It remains to prove the decay for  $I$  and  $\tilde{I}$ . This is done using integration by parts.

From now on  $K_p$  is a generic name for a constant that depends only on  $p$ , and we may use simplification such as  $K_p K_p = K_p$ ,  $const.K_p = K_p$  (but of course only finitely many times...).

The key identity is

$$\left[1 + \frac{f(t)}{\sqrt{\alpha}}\right] e^{-ik_0 t} = \left[1 + i\varepsilon(t) \frac{1}{\sqrt{\alpha}} \frac{\Delta}{\Delta k_0}\right] e^{-ik_0 t} \quad (3.28)$$

where  $\varepsilon(t)$  is the sign of  $\sin(2\pi T t)$  and the discretized derivative  $\frac{\Delta}{\Delta k_0}$  on a function  $F(k_0)$  is defined by

$$\frac{\Delta}{\Delta k_0} F(k_0) = \frac{1}{4\pi T} [F(k_0 + 2\pi T) - F(k_0 - 2\pi T)]. \quad (3.29)$$

Let us consider first the case  $p = 1$ , then we apply this identity inside  $I$  only once. Performing integration by parts we get

$$\begin{aligned} I(\alpha, t) &= \sqrt{\alpha} T \frac{1}{\left(1 + \frac{f(t)}{\sqrt{\alpha}}\right)} \sum_{k_0} e^{-itk_0} \left[1 - i\varepsilon(t) \frac{1}{\sqrt{\alpha}} \frac{\Delta}{\Delta k_0}\right] e^{-\alpha k_0^2} \\ &= \sqrt{\alpha} T \frac{1}{\left(1 + \frac{f(t)}{\sqrt{\alpha}}\right)} \sum_{k_0} e^{-itk_0} \left[1 + i\varepsilon(t) \frac{\sinh 4\pi T \alpha k_0}{2\pi T \sqrt{\alpha}} e^{-\alpha(2\pi T)^2}\right] e^{-\alpha k_0^2}. \end{aligned} \quad (3.30)$$

Now inserting absolute values we have

$$\begin{aligned} |I| \left(1 + \frac{f(t)}{\sqrt{\alpha}}\right) &\leq \sqrt{\alpha} T \sum_{k_0} \left[1 + \frac{|\sinh 4\pi T \alpha k_0|}{2\pi T \sqrt{\alpha}} e^{-\alpha(2\pi T)^2}\right] e^{-\alpha k_0^2} \\ &\leq 2\sqrt{\alpha} T \sum_{k_0 > 0} \left[1 + \frac{\sinh(4\pi T \alpha k_0)}{2\pi T \sqrt{\alpha}} e^{-\alpha(2\pi T)^2}\right] e^{-\alpha k_0^2} \\ &\leq 2\sqrt{\alpha} T \sum_{k_0 > 0} \left[1 + 2\sqrt{\alpha} k_0 e^{4\pi T \alpha k_0} e^{-\alpha(2\pi T)^2}\right] e^{-\alpha k_0^2} \leq const, \end{aligned} \quad (3.31)$$

where in the last line we used that  $\sinh x \leq x e^x$  for all  $x \geq 0$ . For  $p > 1$  we must apply (3.28)  $p$  times:

$$I(\alpha, t) \left(1 + \frac{f(t)}{\sqrt{\alpha}}\right)^p = \sqrt{\alpha} T \sum_{k_0} e^{-itk_0} \left[1 - i\varepsilon(t) \frac{1}{\sqrt{\alpha}} \frac{\Delta}{\Delta k_0}\right]^p e^{-\alpha k_0^2}. \quad (3.32)$$

Each new derivative

- either extracts a new  $\left[1 + i\varepsilon(t)\frac{\sinh 4\pi T\alpha k_0}{2\pi T\sqrt{\alpha}}e^{-\alpha(2\pi T)^2}\right]$  factor from the exponential,
- or applies to a factor derived before.

So we obtain a sum of terms of the following type

$$\begin{aligned}
1) \quad & \left(\frac{\sinh 4\pi T\alpha k_0}{2\pi T\sqrt{\alpha}}\right)^n e^{-n\alpha(2\pi T)^2} \leq (\sqrt{\alpha}k_0)^n e^{n4\pi T\alpha k_0} e^{-n\alpha(2\pi T)^2} \quad \text{for } n \leq p, \\
2) \quad & \left(\frac{1}{\sqrt{\alpha}}\frac{\Delta}{\Delta k_0}\right)^{2n} \frac{\sinh 4\pi T\alpha k_0}{2\pi T\sqrt{\alpha}} = \sinh 4\pi T\alpha k_0 \left(\frac{\sinh 8\pi^2 T^2 \alpha}{2\pi T\sqrt{\alpha}}\right)^{2n-2} \left(\frac{\sinh 8\pi^2 T^2 \alpha}{2\pi T^2 \alpha}\right) \\
& \leq e^{4\pi T\alpha k_0} \text{const} \quad \text{for } 2n \leq p, \\
3) \quad & \left(\frac{1}{\sqrt{\alpha}}\frac{\Delta}{\Delta k_0}\right)^{2n+1} \frac{\sinh 4\pi T\alpha k_0}{2\pi T\sqrt{\alpha}} = \cosh 4\pi T\alpha k_0 \left(\frac{\sinh 8\pi^2 T^2 \alpha}{2\pi T\sqrt{\alpha}}\right)^{2n-1} \left(\frac{\sinh 8\pi^2 T^2 \alpha}{2\pi T^2 \alpha}\right) \\
& \leq e^{4\pi T\alpha k_0} \text{const} \quad \text{for } 2n+1 \leq p.
\end{aligned} \tag{3.33}$$

The first term after summing over  $k_0$  is bounded by  $K_p e^{n(n-1)\alpha(2\pi T)^2}$ . This factor is bounded as  $\alpha T^2 \leq 1$ . In the second and last terms the factors  $\sinh(8\pi^2 T^2 \alpha)/(2\pi T^2 \alpha)$  and  $\sinh(8\pi^2 T^2 \alpha)/(2\pi T\sqrt{\alpha})$  are bounded as long as  $\alpha T^2 \leq 1$ . The same arguments hold for  $\tilde{I}$ .

In the case  $\alpha T^2 > 1$  we cannot go beyond  $p = 1$  as this time the factors  $e^{n(n-1)\alpha(2\pi T)^2}$  are not bounded.  $\square$

### 3.2 Slice decomposition on the propagator

After introducing the Schwinger representation of the propagator we obtained (3.10)-(3.11). Now, fixing a positive number  $M > 1$  we want to cut out as usual  $j_m$  main RG slices following a geometric progression of ratio  $M$ , where  $j_m$  is defined as the temperature scale such that  $M^{j_m} \simeq 1/T$ , more precisely

$$j_m = 1 + \text{Int} \left[ \frac{\ln(T^{-1})}{\ln M} \right] \tag{3.34}$$

where  $\text{Int}$  means the integer part.

### 3.2.1 Heuristic analysis

Remark that  $x$  enters only in the oscillating factor  $e^B$  of the  $\beta$  integral. For  $|x|$  large this integral should be approximated by the region near the saddle points, where  $\partial_\beta B = 0$ , namely  $|\beta| = |x|/2$ . On the other hand,  $t$  enters only in the  $I$  and  $\tilde{I}$  factors. The  $t$  dependence is controlled by the decay of these factors, which is in  $(1 + f(t)/\sqrt{\alpha})^{-p}$  for  $\alpha \leq 1/T^2$ , so the  $\alpha$  integral for  $t$  large should be concentrated around  $\alpha \gtrsim t^2$ , and in fact around  $\alpha \simeq t^2$  (taking into account the  $\alpha^{-2}$  which ensures convergence of the  $\alpha$  integral).

Recall that in [1] the decomposition was done in  $x$  space, with as key relation defining the main slice:

$$M^{j-1} \leq (1 + |x|)^{\frac{3}{4}}(1 + |t| + |x|)^{\frac{1}{4}} \leq M^j . \quad (3.35)$$

For  $|t| \leq |x|$  this relation gives  $|x| \propto M^j$ ,  $|t| \leq M^j$ . For  $|t| > |x|$  an auxiliary decoupling  $|t| \propto M^{j+k}$  was introduced. Then  $|x| \propto M^{j-k/3}$ . We can mimick this slicing by observing that  $\alpha \simeq t^2$ ,  $|\beta| \simeq |x|$ . Hence the slicing relations in parametric space should be

$$M^{2j-2} \leq (1 + \beta^2)^{\frac{3}{4}}(\alpha + \beta^2)^{\frac{1}{4}} \leq M^{2j} , \quad \alpha \simeq M^{2j+2k} . \quad (3.36)$$

However since stationary phase analysis should not be done with sharp boundary to avoid large boundary terms, we have to introduce these relations through smooth rather than sharp cutoff functions in the parametric space.

### 3.2.2 Notation

For any two real numbers  $X, Y$  we will write  $X \leq_c Y$  if there is a constant  $1/10 < C < 10$  such that  $X \leq CY$ . The same holds for  $X \geq_c Y$  and  $X \simeq Y$ .

### 3.2.3 The slicing

Motivated by this heuristic discussion we write the propagator  $C = \sum_{j=0}^{j_m} C^j$ , where

$$C^j(t, x) = \int_1^\infty d\alpha \int_{\mathbb{R}} d\beta \chi^j(X_{\alpha\beta}) F(\alpha, \beta, t, \vec{x}) \quad (3.37)$$

where

$$X_{\alpha\beta} = (1 + \beta^2)^{\frac{3}{4}}(\alpha + \beta^2)^{\frac{1}{4}}, \quad (3.38)$$

and

$$\begin{aligned}
\chi_j(X) &= u\left(\frac{X}{M^{2j}}\right) - u\left(\frac{X}{M^{2j-2}}\right) & j_m > j > 0 \\
\chi_0(X) &= u(X) \ , & j = 0 \\
\chi_{j_m}(X) &= 1 - u\left(\frac{X}{M^{2j_m-2}}\right) \ , & j = j_m
\end{aligned} \tag{3.39}$$

and  $u(x)$  is a smooth function with compact support such that  $u(x) = 1$  for  $0 \leq x \leq 1$  and  $u(x) = 0$  for  $x > 2$ . Note that this definition implies the following constraints:

$$\begin{aligned}
\chi_j(X_{\alpha\beta}) \neq 0 &\Rightarrow X_{\alpha\beta} \simeq M^{2j} \\
\chi_0(X_{\alpha\beta}) \neq 0 &\Rightarrow X_{\alpha\beta} \leq_c 1 \\
\chi_{j_m}(X_{\alpha\beta}) \neq 0 &\Rightarrow X_{\alpha\beta} \geq_c M^{2j_m}
\end{aligned} \tag{3.40}$$

where in the first line we took  $j_m > j > 0$ .

**Remark** Actually, in [1] we introduced  $j_m + 1$  scales, but all the estimates we obtained remain valid with  $j_m$  scales.

Now, as in [1] we distinguish two situations.

- If  $\alpha \leq \beta^2$ , then  $\chi_j(X) \neq 0$  only for  $\beta \simeq M^j$ . For the last scale  $j = j_m$  we will have to distinguish the case  $\alpha \leq 1/T^2 = M^{2j_m}$  and the case  $M^{2j_m} < \alpha \leq \beta^2$ .
- If  $\alpha \geq \beta^2$ , we have to add an auxiliary decomposition over the possible size of  $\alpha$ .

### 3.2.4 Auxiliary scales

As in [1] for each  $j \leq j_m$  we add the decomposition

$$1 = \sum_{k=0}^{k_m(j)} \tilde{\chi}_k(\alpha) \ , \tag{3.41}$$

where for  $k_m(j) > 0$  we define

$$\begin{aligned}\tilde{\chi}_k(\alpha) &= u\left(\frac{\alpha}{M^{2j+2k}}\right) - u\left(\frac{\alpha}{M^{2j+2k-2}}\right) \quad \text{for } k_m(j) > k \geq 1, \\ \tilde{\chi}_0(\alpha) &= u\left(\frac{\alpha}{M^{2j}}\right), \quad k = 0 \\ \tilde{\chi}_{k_m(j)}(\alpha) &= 1 - u\left(\frac{\alpha}{M^{2j+2k_m(j)}}\right), \quad k = k_m(j),\end{aligned}\tag{3.42}$$

and for  $k_m = 0$  we have no decomposition:

$$\tilde{\chi}_0(\alpha) = 1. \tag{3.43}$$

Finally, as in [1]  $k_m(j)$  is defined as

$$k_m(j) = \min[(j_m - j), 3j]. \tag{3.44}$$

These definitions imply the following constraints on  $\alpha$  (when  $k_m(j) > 0$ ):

$$\begin{aligned}\tilde{\chi}_k(\alpha) \neq 0 &\Rightarrow \alpha \simeq M^{2j+2k}, \\ \tilde{\chi}_0(\alpha) \neq 0 &\Rightarrow 1 \leq \alpha \leq_c M^{2j}, \\ \tilde{\chi}_{k_m(j)}(\alpha) \neq 0 &\Rightarrow \alpha \geq_c M^{2j+2k_m},\end{aligned}\tag{3.45}$$

where in the first line we take  $k_m(j) > k > 0$ .

**Remark** In [1] the bound  $k \leq 3j$  was obtained observing that  $f(t)^{1/4} \leq M^j$  for  $j \leq j_m$ , while the bound  $k \leq j_m - j$  was due to  $f(t) \leq M^{j_m}$  (by definition of  $f(t)$ ). Here the first bound is still valid since  $\alpha^{1/4} \leq M^{2j}$ , but the second no longer holds since  $\alpha$  can take any value in  $[1, \infty)$ . Nevertheless we take the same definition of  $k_m(j)$  as in [1] so that the results we obtained there can be directly applied here.

The slicing of the propagator is then

$$C = \sum_{j=0}^{j_m} \sum_{k=0}^{k_m(j)} C^{jk}, \tag{3.46}$$

where

$$C^{jk}(t, x) = \int_1^\infty d\alpha \int_{\mathbb{R}} d\beta \chi_{jk}(\alpha, \beta) F(\alpha, \beta, t, \vec{x}), \tag{3.47}$$

$F(\alpha, \beta, t, \vec{x})$  was introduced in (3.13)-(3.14) and we defined

$$\chi_{jk}(\alpha, \beta) = \chi^j(X_{\alpha\beta}) \tilde{\chi}^k(\alpha) . \quad (3.48)$$

Note that this slicing selects  $\alpha \geq \beta^2$  when  $k > 0$  and  $\alpha \leq \beta^2$  when  $k = 0$ . This is proved in the following lemma.

**Lemma 2** *If  $0 < j < j_m$  then  $k_m > 0$  and:*

- *for all  $k_m \geq k > 0$  we have  $\chi_{jk} \neq 0 \Rightarrow \alpha \geq_c \beta^2$  and*
- *for  $k = 0$  we have  $\chi_{jk} \neq 0 \Rightarrow \alpha \leq_c \beta^2$ .*

**Proof** Since  $0 < j < j_m$ ,  $k_m > 0$  by definition and  $\chi_j \neq 0$  implies  $X_{\alpha\beta} \sim M^{2j}$ .

Let  $k > 0$  and suppose  $1 \leq \alpha < \beta^2$ . Then  $X_{\alpha\beta} \sim \beta^2 \sim M^{2j}$ . Moreover, for all  $k > 0$   $\tilde{\chi}_k \neq 0$  implies  $\alpha \geq M^{2j+2k}$ . So  $M^{2j+2k} \leq \alpha < M^{2j}$ , which is impossible.

Let  $k = 0$  and suppose  $\alpha > \beta^2 \geq 1$ . Then  $X_{\alpha\beta} \sim \alpha^{1/4}(\beta^2)^{3/4}$  and  $(\beta^2)^{3/4} \sim M^{2j}/\alpha^{1/4}$ . Moreover  $\tilde{\chi}_0 \neq 0$  implies  $1 \leq \alpha \leq M^{2j}$  so  $M^{2j} \geq \alpha > \beta^2 \geq M^{2j}$ . That's impossible unless  $\alpha \sim \beta^2$ .

Finally let  $k = 0$  and suppose  $\alpha > \beta^2$  and  $\beta^2 \leq 1$ . Then  $X_{\alpha\beta} \sim \alpha^{1/4} \sim M^{2j}$ . But  $\tilde{\chi}_0 \neq 0$  implies  $1 \leq \alpha \leq M^{2j}$  so  $M^{2j} \geq \alpha \sim M^{8j}$ . That's impossible. The result follows.  $\square$

Now we distinguish three cases.

**Case 1** For  $j = 0$  we have  $k_m = 0$  (no auxiliary scales) and

$$\chi_{jk}(\alpha, \beta) = \chi_j(X_{\alpha\beta}) \neq 0 \quad \Rightarrow \quad \alpha \simeq 1, \beta^2 \leq 1 . \quad (3.49)$$

**Case 2** For  $0 < j < j_m$  we have  $k_m \geq 1$  and  $\chi_{jk}(\alpha, \beta) \neq 0 \Rightarrow$

- a)  $0 < k < k_m$ : then

$$\alpha \simeq M^{2j+2k}, |\beta| \simeq M^{j-k/3} \quad (3.50)$$

- b)  $k = 0$ : then

$$1 \leq \alpha \leq M^{2j}, |\beta| \simeq M^j \quad (3.51)$$

- c)  $k = k_m$  and  $\beta^2 \leq 1$ : then

$$\alpha \simeq M^{8j} , \quad 0 \leq \beta^2 \leq 1 \quad (3.52)$$

- d)  $k = k_m$  and  $\beta^2 > 1$ : then  $k_m = j_m - j$ , which means  $4j \geq j_m$  and

$$M^{2j_m} \leq \alpha \leq M^{8j} , \quad \beta^2 \simeq \frac{M^{8j/3}}{\alpha^{1/3}} \quad (3.53)$$

Note that in this last case we have  $1 \leq |\beta| \leq M^{(4j-j_m)/3}$

**Case 3** Finally for  $j = j_m$  we have  $k_m = 0$  and  $\chi_{j_m 0}(\alpha, \beta) \neq 0 \Rightarrow$  one of the following three situations holds:

$$\begin{aligned} a) \quad & M^{8j_m} < \alpha < \infty , \quad 0 \leq \beta^2 \leq 1 \\ b) \quad & M^{2j_m} \leq \alpha \leq M^{8j_m} , \quad \frac{M^{8j_m/3}}{\alpha^{1/3}} \leq \beta^2 \leq \alpha \\ c) \quad & 1 \leq \alpha \leq \beta^2 , \quad M^{2j_m} \leq \beta^2 < \infty \end{aligned} \quad (3.54)$$

The typical situation is Case 2a and 2b, that is  $0 < j < j_m$  and  $0 \leq k < k_m$ .

## 4 Scaled Decay

**Theorem 1** *Let  $C^{jk}$  be the scaled propagator introduced in (3.47). Then for any  $j < j_m, 0 \leq k \leq k_m$  and  $p > 0$  the decay is*

$$|C^{jk}(t, x)| \leq K_p \frac{M^{-2j-2k/3}}{[(1 + f(t)M^{-j-k})(1 + |x|M^{-j+k/3})]^p} \quad (4.55)$$

where  $K_p$  is a constant dependent on  $p$ . For the last scale  $j = j_m$  we have no decay in  $t$  at all and  $x$  decay according to the infrared cutoff  $T$

$$|C^{j_m 0}(t, x)| \leq K_p \frac{T^2}{(1 + |x|T)^p} = K_p \frac{M^{-2j_m}}{(1 + |x|M^{-j_m})^p} \quad (4.56)$$

Remark that this decay is identical to the one of [1], section II.5 (equations II.43, II.45 and II.47)

**Proof** The rest of this section is devoted to the proof. We treat separately the cases listed in sect 3.2.3.

#### 4.1 Case 1: $\alpha \simeq 1, \beta^2 \leq 1$

This corresponds to  $j = k = 0$  so we need to prove

$$|C^{00}(t, x)| \leq K_p \frac{1}{[(1 + f(t))(1 + |x|)]^p} . \quad (4.57)$$

Since  $\alpha \simeq 1$   $I$  and  $\tilde{I}$  are both bounded by  $K (1 + f(t))^{-p}$  (using (3.17)) thus giving the  $t$  decay. To perform the  $\beta$  integral we distinguish two cases:

a) When  $|x| > 1$  we need to extract the spatial decay. We apply

$$e^{-ix^2/(4\beta)} = -i \frac{4\beta^2}{x^2} \frac{d}{d\beta} e^{-ix^2/(4\beta)} \quad (4.58)$$

We perform integration by parts in  $\beta$  several times. Then we obtain the decay  $|x|^{-2p}$ . Now we can insert absolute values in the  $\beta$  integral that is now bounded by a constant, since  $\beta \leq 1$ . Note that the additional  $\beta$  factor we obtain from integration by parts ensures the integral over  $\beta$  has no divergence in  $\beta = 0$ .

b) When  $|x| \leq 1$  we do not need to extract any spatial decay. We only need to prove the integral over  $\beta$  is bounded. To avoid the  $\beta^{-3/2}$  divergence we go back to (3.13) and after performing several times integration by parts on

$$e^{i\beta(k^2-1)} = \frac{1}{1 + i(k^2 - 1)} \left( 1 + \frac{d}{d\beta} \right) e^{i\beta(k^2-1)} \quad (4.59)$$

we can insert absolute values in the  $k$  and  $\beta$  integral. Then no divergence in  $\beta$  appears.

#### 4.2 Case 2a: $\alpha \simeq M^{2j+2k}, |\beta| \simeq M^{j-k/3}$

This corresponds to  $0 < j < j_m, 0 < k < k_m$ . The  $\beta$  integral is performed through a saddle analysis. The saddle point for the phase factor  $B$  (3.15) is  $\beta_s = \pm|x|/2$ . Therefore we introduce a smooth decomposition

$$1 = \eta(Y) + (1 - \eta(Y)) , \quad Y = \frac{(\beta - \beta_s)}{\beta_s} , \quad \beta_s = \pm \frac{|x|}{2} , \quad x \neq 0$$



where  $\eta$  has compact support  $\eta \neq 0$  if  $Y < 1/10$ . As

$$\partial_\beta B(\beta, x) = 1 - \frac{1}{\left(1 + \frac{\beta - \beta_s}{\beta_s}\right)^2},$$

it is not difficult to see that inside the support of  $1 - \eta$  we have  $|\partial_\beta B(\beta, x)| \geq 1/200$ .

#### 4.2.1 Saddle region

This region comes into play only when the support of  $\eta$  has a non empty interaction with the support  $\chi^{jk}$  (in this case that means  $|x| \simeq M^{j-k/3}$ ). Then we have to study

$$I_s = \int_1^\infty d\alpha \int_{\mathbb{R}} d\beta \chi_{jk}(\alpha, \beta) \eta\left(\frac{\beta - \beta_s}{\beta_s}\right) \frac{\tilde{I}(\alpha, t) + i\beta I(\alpha, t)}{\alpha^2 \beta^{3/2}} e^{-\frac{\beta^2}{4\alpha}} e^{-iB(\beta, x)}. \quad (4.60)$$

Near the positive saddle  $\beta_s = |x|/2$  we have

$$\begin{aligned} B(\beta) &= B(\beta_s) + \frac{2}{|x|} (\beta - \beta_s)^2 + O\left(\frac{(\beta - \beta_s)^3}{|x|^2}\right) \\ &= B(\beta_s) + (\beta - \beta_s)^2 \frac{2}{|x|} [1 + R(\beta - \beta_s)], \\ R(\beta - \beta_s) &\propto \frac{|\beta - \beta_s|}{|x|} < 1/10. \end{aligned} \quad (4.61)$$

As we have a phase factor, it is not easy to perform the Gaussian integral in  $y = \beta - \beta_s$ , but we know the result should be  $\sqrt{|x|} = M^{(j-k/3)/2}$ . In order to prove that  $|y| \leq \sqrt{|x|}$  we perform integration by parts in the following way:

$$e^{-iB(\beta, x)} = \frac{1}{(1 - i\sqrt{x} \partial_\beta B(\beta, x))} \left(1 + \sqrt{x} \frac{\partial}{\partial \beta}\right) e^{-iB(\beta, x)}. \quad (4.62)$$

As  $-iB$  is a phase factor,  $-i\partial_\beta B$  is pure imaginary so the denominator is well defined. Now for  $|y| = |\beta - \beta_s| \leq |x|$  we have

$$\partial_\beta B(\beta, x) = 4 \frac{y}{|x|} [1 + R(y)] \quad \text{with} \quad |R(y)| \propto \frac{|y|}{|x|} < 1/10.$$

Therefore  $|\partial_\beta B(\beta, x)| \geq \frac{|y|}{|x|}$  and

$$\left| \frac{1}{(1 + \sqrt{x} \partial_\beta B)} \right| \leq \frac{1}{(1 + \frac{|y|}{\sqrt{x}})}. \quad (4.63)$$

Performing integration by parts with respect to  $\beta$  we get

$$I_s = \int_1^\infty d\alpha \int_{\mathbb{R}} d\beta \frac{e^{-\frac{\beta^2}{4\alpha}} e^{-iB(\beta, x)}}{(1 - i \sqrt{x} \partial_\beta B)} \frac{\tilde{I}(\alpha, t) + i\beta I(\alpha, t)}{\alpha^2 \beta^{3/2}} \quad (4.64)$$

$$\left[ \chi_{jk}(\alpha, \beta) \eta \left( \frac{\beta - \beta_s}{\beta_s} \right) \left( 1 - i \frac{|x| \partial_\beta^2 B}{(1 - i \sqrt{|x|} \partial_\beta B)} \right) + R(\alpha, \beta) \right]$$

where

$$|R(\alpha, \beta)| \leq O \left( \frac{\sqrt{|x|}}{\beta} \right) \bar{\chi}_{jk}(\alpha, \beta) \bar{\eta} \left( \frac{\beta - \beta_s}{\beta_s} \right) \quad (4.65)$$

and  $\bar{\chi}_{jk}$ ,  $\bar{\eta}$  have a slightly larger support than  $\chi_{jk}$ ,  $\eta$ .

Repeating once and inserting absolute values inside the integral we obtain

$$|I_s| \leq \int_1^\infty \frac{d\alpha}{\alpha^2} \int_{\mathbb{R}} \frac{d\beta}{|\beta|^{3/2}} \frac{\bar{\chi}_{jk}(\alpha, \beta) \bar{\eta} \left( \frac{\beta - \beta_s}{\beta_s} \right)}{\left( 1 + \frac{|\beta - \beta_s|}{\sqrt{|x|}} \right)^2} e^{-\frac{\beta^2}{4\alpha}} (|\tilde{I}| + |\beta| |I|) \quad (4.66)$$

where we applied (4.63) and  $\sqrt{|x|}/\beta < 1$ . We bound  $\beta e^{-\frac{\beta^2}{4\alpha}} \leq \sqrt{\alpha} e^{-\frac{\beta_s^2}{4\alpha}}$ . Using (3.17) to bound  $I$  and  $|\tilde{I}|$  and  $\alpha \simeq M^{j+k}$  we get

$$\begin{aligned} |I_s| &\leq \frac{K_p}{(1 + f(t)M^{-j-k})^p} \int_1^{\frac{1}{T}} \frac{d\alpha}{\alpha^2} \sqrt{\alpha} \int_{-\infty}^\infty \frac{d\beta}{\beta_s^{3/2}} \frac{\bar{\chi}_{jk}(\alpha, \beta_s) \bar{\eta} \left( \frac{\beta - \beta_s}{\beta_s} \right)}{\left( 1 + \frac{|\beta - \beta_s|}{\sqrt{|x|}} \right)^2} e^{-\frac{\beta_s^2}{4\alpha}} \\ &\leq \frac{K_p}{(1 + f(t)M^{-j-k})^p} \int_{\alpha \simeq M^{2j+2k}} \frac{d\alpha}{\alpha^{3/2}} \int_{-\infty}^\infty \frac{dy}{\left( 1 + \frac{|y|}{\sqrt{|x|}} \right)^2} \frac{1}{|x|^{3/2}} \\ &\leq \frac{K_p}{(1 + f(t)M^{-j-k})^p} \frac{M^{2j+2k}}{M^{3j+3k}} \frac{\sqrt{|x|}}{|x|^{3/2}} \\ &= \frac{K_p}{(1 + f(t)M^{-j-k})^p} M^{-2j-\frac{2}{3}k}. \end{aligned} \quad (4.67)$$

In the first line  $\bar{\eta}$  ensures  $\beta \simeq \beta_s = |x| \simeq M^{j-k/3}$ . As  $|x| \simeq M^{j-k/3}$  we do not need to gain any further spatial decay hence Lemma 1 holds in this case.

### 4.2.2 Region far from the saddle

This region comes into play only when the support of  $1 - \eta$  has a non empty interaction with the support  $\chi^{jk}$ . Then we have to study

$$I_f = \int_1^\infty d\alpha \int_{\mathbb{R}} d\beta \chi_{jk}(\alpha, \beta) \left[ 1 - \eta\left(\frac{\beta - \beta_s}{\beta_s}\right) \right] \frac{(\tilde{I}(\alpha, t) + i\beta I(\alpha, t))}{\alpha^2 \beta^{3/2}} e^{-\frac{\beta^2}{4\alpha}} e^{-iB(\beta, x)}. \quad (4.68)$$

In this region  $\partial_\beta B > 1/200$  so we can apply

$$e^{-iB(\beta, x)} = \frac{1}{\partial_\beta B} \frac{\partial}{\partial \beta} e^{-iB}. \quad (4.69)$$

Performing integration by parts  $p + 1$  times and inserting absolute values we get

$$|I_f| \leq K_p \int d\alpha \int d\beta \bar{\chi}_{jk}(1 - \bar{\eta}_1) e^{-\frac{\beta^2}{4\alpha}} e^{-iB} \frac{(|\tilde{I}| + |\beta||I|)}{\alpha^2 |\beta|^{3/2}} \frac{1}{|\partial_\beta B|^{p+1}} O\left(\frac{1}{|\beta|^{p+1}}\right). \quad (4.70)$$

where  $\bar{\eta}_1$  has slightly smaller support than  $\eta$  (and  $\bar{\chi}$  is the same as in the saddle region) Note that when the derivative hits  $\eta$  instead of a  $1/\beta$  we get a  $1/\beta_s$  factor. But

$$\frac{1}{\beta_s} \eta' \leq \text{const}(1 - \bar{\eta}_1) \frac{1}{|\beta|}. \quad (4.71)$$

Since  $\partial_\beta B = 1 - |x|^2/4\beta^2$ , the  $1 - \bar{\eta}_1$  function ensures that  $||2\beta/x| - 1| \geq \text{const}$ , and  $|\beta| \simeq M^{j-k/3} \geq 1$  we have

$$\frac{1}{|\partial_\beta B|^{p+1}} \leq \frac{K}{(1 + |x|M^{-j-k/3})^p} \quad (4.72)$$

The factor  $\beta^{-1-p}$  ensures the global factor is correct. Actually we need only to use  $\beta^{-1}$ :

$$\begin{aligned} |I_f| &\leq \frac{K_p}{(1 + |x|^2/M^{j-k/3})^p} \frac{1}{(1 + f(t)M^{-j-k})^p} \int \frac{d\alpha}{\alpha^{3/2}} \int \frac{d\beta}{\beta^{5/2}} \bar{\chi}_{jk} \\ &\leq \frac{K_p}{(1 + |x|^2/M^{j-k/3})^p} \frac{1}{(1 + f(t)M^{-j-k})^p} M^{-j-k} M^{-(3/2)(j-k/3)} \\ &\leq \frac{K_p}{(1 + |x|^2/M^{j-k/3})^p} \frac{1}{(1 + f(t)M^{-j-k})^p} M^{-2j-\frac{2}{3}k} \end{aligned} \quad (4.73)$$

as  $j - k/3 \geq 0$ .

### 4.3 Case 2b: $1 \leq \alpha \leq M^{2j}, |\beta| \simeq M^j$

This corresponds to  $k = 0, 0 < j < j_m$  so we need to prove

$$|C^{j0}(t, x)| \leq K_p \frac{M^{-2j}}{[(1 + f(t)M^{-j})(1 + |x|M^{-j})]^p} \quad (4.74)$$

This case is treated exactly as the  $k > 0$  case. The only difference is that now  $\alpha$  has no fixed value, but instead must be integrated between 1 and  $\beta^2$ . This can be done using the exponential decay (after performing all the necessary integration by parts in the  $\beta$  integral)

$$\int_1^{\beta^2} \frac{d\alpha}{\alpha} \frac{1}{\sqrt{\alpha}} e^{-\frac{\beta^2}{\alpha}} = \frac{1}{\beta} \int_{\beta^{-2}}^1 \frac{d\alpha}{\alpha} \frac{1}{\sqrt{\alpha}} e^{-\frac{1}{\alpha}} \leq \frac{K}{\beta} \simeq \frac{K}{M^j} \quad (4.75)$$

where  $K$  is some constant. As a consequence the  $\alpha$  integral is bounded by  $M^{-j}$ . The  $t$  decay can be obtained observing that as  $\alpha \leq \beta^2$  then

$$\frac{1}{\left(1 + \frac{f(t)}{\sqrt{\alpha}}\right)^p} \leq \frac{1}{\left(1 + \frac{f(t)}{\beta}\right)^p} \simeq \frac{1}{\left(1 + \frac{f(t)}{M^j}\right)^p}. \quad (4.76)$$

The  $x$  decay is treated by the same saddle/off-saddle analysis as for  $k > 0$ .

### 4.4 Case 2c: $\alpha \simeq M^{8j}, \beta^2 \leq 1$

This corresponds to  $0 < j < j_m$  and  $k = k_m$  so we must distinguish between the two possible values for  $k_m$ .

(i) If  $k_m = 3j$ , then  $4j \leq j_m$  and we must prove

$$|C^{jk_m}(t, x)| \leq K_p \frac{M^{-4j}}{[(1 + f(t)M^{-4j})(1 + |x|)]^p} \quad (4.77)$$

Since  $\beta^2 \leq 1$ , we treat the  $\beta$  integral (and  $|x|$  decay) as in Case 1 (distinguish  $|x| > 1$  and  $|x| \leq 1$ ). The bounds on  $I$  and  $\tilde{I}$  from (3.17) give the correct decay in  $t$  and the  $\alpha$  integral gives the prefactor  $M^{-4j}$ .

(ii) If  $k_m = j_m - j$ , then  $4j \geq j_m$  and we want to prove

$$|C^{jk}(t, x)| \leq K_p \frac{M^{-4j} M^{2/3(4j-j_m)}}{[(1 + f(t)M^{-j_m})(1 + |x|M^{-(4j-j_m)/3})^p]} \quad (4.78)$$

Since  $4j \geq j_m$  we have  $\alpha \simeq M^{8j} \geq M^{2j_m} = 1/T^2$ . Therefore the bound on  $I, \tilde{I}$  from (3.17) give the correct decay in  $t$ . For  $|x|$ , repeating the same arguments as in Case 1, it is easy to get a decay  $(1 + |x|)^{-p} \leq (1 + |x|M^{-(4j-j_m)/3})^{-p}$  since  $4j - j_m \geq 0$ . Finally the  $\alpha$  integral is bounded by  $M^{-4j} \leq M^{-4j} M^{2/3(4j-j_m)}$ .

#### 4.5 Case 2d: $M^{2j_m} \leq \alpha \leq M^{8j}$ , $\beta^2 \simeq \frac{M^{8j/3}}{\alpha^{1/3}}$

This corresponds to  $0 < j < j_m$  and  $k = k_m$  with  $k_m = j_m - j$ . and we need to prove

$$|C^{jk_m}(t, x)| \leq K_p \frac{M^{-4j} M^{2/3(4j-j_m)}}{[(1 + |x|M^{-(4j-j_m)/3})^p]} . \quad (4.79)$$

Note that in this case there is no  $t$  decay. Since  $M^{2j_m} \leq \alpha$  the bounds (3.18) on  $I, \tilde{I}$  give the correct decay in  $t$ . The  $\beta$  integral is performed by a saddle analysis (region near/far from the saddle) as in Case 2a.

The result is the correct decay in  $x$  times a factor  $(\sqrt{\alpha} + \beta_s)\sqrt{\beta_s}/(\alpha^2 \beta_s^{3/2})$ , where we used  $\beta^2 \leq \alpha$ . Inserting the value of  $\beta_s^2 = M^{8j/3}/\alpha^{1/3}$  and performing the  $\alpha$  integral we get the correct prefactor.

#### 4.6 Case 3

The last case corresponds to  $j = j_m$ , so  $k_m = 0$ , so we want to prove

$$|C^{j_m 0}(t, x)| \leq K_p \frac{M^{-2j_m}}{(1 + |x|M^{-j_m})^p} \quad (4.80)$$

We have three possible ranges for  $\alpha$  and  $\beta$ .

##### 4.6.1 Case 3.a

$M^{8j} \leq \alpha \leq \infty$ ,  $\beta^2 \leq 1$ . Since  $\alpha > M^{2j_m}$  we apply (3.18) so

$$|I + i\beta\tilde{I}| \leq K(T^{-1} + \beta)(1 + f(t)T)^{-1} .$$

Since  $\beta^2 \leq 1$ , by applying the same  $|x| > 1/|x| \leq 1$  analysis as in Case 1 we obtain

$$\int d\beta \chi_{jk}(\alpha, \beta) F(\alpha, \beta, t, \vec{x}) \leq \frac{1}{\alpha^2} \frac{K_p M^{j_m}}{(1 + f(t)T) (1 + |x|)^p} \leq \frac{K_p}{\alpha^2} \frac{1}{(1 + |x|)^p}$$

The  $\alpha$  integral is then bounded by  $M^{-8j}$  so

$$|C^{j_m 0}(t, x)| \leq \frac{K_p M^{-8j_m}}{(1 + |x|)^p} < \frac{K_p M^{-2j_m}}{(1 + |x| M^{-j_m})^p}.$$

#### 4.6.2 Cases 3.b and 3.c

To get the  $|x|$  decay we use the infrared cutoff on  $k_0$  ( $|k_0| \geq T$ ) (as we did in [1] equation II.9). Let consider first the case  $|x| > 1$ .

Inside the  $\beta$  integral we apply the identity:

$$e^{-\frac{|x|^2}{4\beta}} = \frac{4\beta^2}{|x|^2} \frac{\partial}{\partial \beta} e^{-\frac{|x|^2}{4\beta}} \quad (4.81)$$

Performing integration by parts with respect to  $\beta$   $p$  times we obtain a factor of order

$$\left( \frac{\beta^2}{|x|^2} \right)^p = \frac{1}{(T|x|)^{2p}} \left( \frac{\beta^2}{\alpha} \right)^p (\alpha T^2)^p \quad (4.82)$$

The first term is exactly the decay we are looking for. The integral over  $\beta$  is now performed using the saddle analysis. The factor  $(\beta^2/\alpha)^p$  is bounded using a piece of the exponential decay  $e^{-\frac{\beta^2}{4\alpha}}$  (after performing the necessary integrations by parts in the  $\beta$  integral). The factor  $(\alpha T^2)^p$  is bounded by the  $e^{-k_0^2 \alpha}$  in  $I, \tilde{I}$ :

$$(\alpha T^2)^p |I| \leq \sqrt{\alpha} T \sum_{k_0} (\alpha T^2)^p e^{-k_0^2 \alpha} \leq c_p \quad (4.83)$$

$$(\alpha T^2)^p |\tilde{I}| \leq \sqrt{\alpha} T \sum_{k_0} (\alpha T^2)^p 2\alpha |k_0| e^{-k_0^2 \alpha} \leq \sqrt{\alpha} c'_p \quad (4.84)$$

$$(4.85)$$

where we used the infrared cutoff  $|k_0| \geq T$ .

The saddle analysis is performed as in Case 2a (see (4.66)-(4.70))

$$\begin{aligned}
|I_s| &\leq \frac{K_p}{(T|x|)^{2p}} \int \frac{d\alpha}{\alpha^2} \int \frac{d\beta}{|\beta|^{3/2}} \frac{\bar{\chi}_{jm0}(\alpha, \beta) \bar{\eta}\left(\frac{\beta - \beta_s}{\beta_s}\right)}{\left(1 + \frac{|\beta - \beta_s|}{\sqrt{|x|}}\right)^2} (\sqrt{\alpha} + |\beta|) e^{-\frac{\beta^2}{8\alpha}} \\
|I_f| &\leq \frac{K_p}{(T|x|)^{2p}} \int \frac{d\alpha}{\alpha^2} \int \frac{d\beta}{|\beta|^{3/2}} \frac{\bar{\chi}_{jk}(1 - \bar{\eta}_1)}{|\beta|} (\sqrt{\alpha} + |\beta|) e^{-\frac{\beta^2}{8\alpha}}.
\end{aligned} \tag{4.86}$$

where we bounded  $|B'|^{-1} \leq \text{const}$  since we already have the  $x$  decay we need. Since  $I_f$  is done in the same way as  $I_s$  (and the bounds are easier) we will only look at  $I_s$ . This last appears only when  $|x|$  belongs to the integration interval for  $\beta$ .

(b) In this case we first perform the  $\beta$  integral. Since  $\beta \leq \sqrt{\alpha}$  we have  $(\sqrt{\alpha} + |\beta|) \leq 2\sqrt{\alpha}$ . The result is  $\sqrt{|x|}/|x|^{3/2}$ , then

$$|I_s| \leq \frac{K_p}{(T|x|)^{2p}} \int \frac{d\alpha}{\alpha^2} \frac{\sqrt{\alpha}}{|x|} \leq \frac{K_p}{(T|x|)^{2p}} \int_{M^{2jm}}^{M^{8jm}} \frac{d\alpha}{\alpha^{3/2}} \frac{\alpha^{1/6}}{M^{4jm/3}} \leq K_p \frac{M^{-2jm}}{(T|x|)^{2p}} \tag{4.87}$$

(c) In this case we first perform the  $\alpha$  integral. Since  $\beta \geq \sqrt{\alpha}$  we have  $(\sqrt{\alpha} + |\beta|) \leq 2|\beta|$ . We perform the  $\alpha$  integral as in (4.75):

$$\int_1^{\beta^2} \frac{d\alpha}{\alpha^2} e^{-\beta^2/\alpha} \leq \frac{K}{\beta^2}. \tag{4.88}$$

The  $\beta$  integral is then bounded by

$$|I_s| \leq \frac{K_p}{(T|x|)^{2p}} \frac{|x|}{|x|^2} \frac{|x|^{1/2}}{|x|^{3/2}} \leq \frac{K_p}{(T|x|)^{2p}} \frac{1}{|x|^2} \leq K_p \frac{M^{-2jm}}{(T|x|)^{2p}} \tag{4.89}$$

since  $|x| \geq M^{jm}$  (we are in the saddle region).

Remark that when  $|x| \leq 1$  we can repeat the bounds above without extracting any  $x$  decay.  $\square$

## References

- [1] M. Disertori, J. Magnen and V. Rivasseau, Interacting Fermi liquid in three dimensions at finite temperature: Part I: Convergent Contributions, Annales Henri Poincaré Vol. 2, 733-806 (2001), arXiv:cond-mat/0012270

- [2] Joel Feldman, Horst Knörrer and Eugene Trubowitz, *Commun. Math. Phys.* **247** (2004): A Two Dimensional Fermi Liquid. Part 1: Overview, 1-47; Part 2: Convergence, 49-111; Part 3: The Fermi Surface, 113-177; Particle-Hole Ladders, 179-194; Convergence of Perturbation Expansions in Fermionic Models. Part 1: Nonperturbative Bounds, 195-242; Part 2: Overlapping Loops, 243-319.
- [3] V. Pasquier, Quantum Hall Effect and Noncommutative Geometry; A. Polychronakos, Noncommutative Fluids; V. Rivasseau, Noncommutative Renormalization, in “Quantum Spaces”, PMP 53, Birkhäuser Verlag (2007).
- [4] M. Salmhofer, Continuous renormalization for Fermions and Fermi liquid theory, *Commun. Math. Phys.* **194**, 249 (1998).
- [5] G. Benfatto and G. Gallavotti, Perturbation theory of the Fermi surface in a quantum liquid. A general quasi particle formalism and one dimensional systems, *Journ. Stat. Phys.* **59** 541 (1990).
- [6] G. Benfatto, G. Gallavotti, A. Procacci and B. Scoppola, *Commun. Math. Phys.* **160**, 93 (1994).
- [7] F. Bonetto and V. Mastropietro, *Commun. Math. Phys.* **172**, 57 (1995).
- [8] M. Disertori and V. Rivasseau, Interacting Fermi liquid in two dimensions at finite temperature, Part I: Convergent Attributions, *Commun. Math. Phys.* **215**, 251 (2000); Part II: Renormalization, in two dimensions at finite temperature, *Commun. Math. Phys.* **215**, 291 (2000).
- [9] G. Benfatto, A. Giuliani and V. Mastropietro, Low temperature analysis of two dimensional Fermi systems with symmetric Fermi surface, *Annales Henry Poincaré*, **4**, 137-193 (2003); G. Benfatto, A. Giuliani and V. Mastropietro, Fermi Liquid Behavior in the 2D Hubbard Model at Low Temperatures, *Ann. H. Poincaré* **7**, 809-898 (2006).
- [10] V. Rivasseau, The two dimensional Hubbard Model at half-filling: I. Convergent Contributions, *Journ. Stat. Phys.* Vol **106**, 693-722 (2002); S. Afchain, J. Magnen and V. Rivasseau, Renormalization of the 2-point function of the Hubbard Model at half-filling, *Ann. Henri Poincaré* **6**, 399, (2005); The Hubbard Model at half-filling, part III: the lower bound on the self-energy, *Ann. Henri Poincaré* **6**, 449 (2005)



- [11] J. Magnen and V. Rivasseau, A Single Scale Infinite Volume Expansion for Three Dimensional Many Fermion Green's Functions, Math. Phys. Electronic Journal, Volume 1, 1995.
- [12] A. Lesniewski, "Effective Action for the Yukawa(2) Quantum Field Theory," Commun. Math. Phys. **108**, 437 (1987).
- [13] A. Abdesselam and V. Rivasseau, "Explicit fermionic tree expansions," Lett. Math. Phys. **44**, 77 (1998).
- [14] V. Rivasseau, "Introduction to the Renormalization Group with Applications to Non-Relativistic Quantum Electron Gases," arXiv:1102.5117 [math-ph].